Hybrid Method for Solving Special Fourth Order Ordinary Differential Equations

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ABSTRACT

In recent time, Runge-Kutta methods that integrate special fourth order ordinary differential equations (ODEs) directly are proposed to address efficiency issues associated with classical Runge-Kutta methods. Although, the methods require approximation of y', y'' and y''' of the solution at every step. In this paper, a hybrid type method is proposed, which can directly integrate special fourth order ODEs. The method does not require the approximation of any derivatives of the solution. Algebraic order conditions of the methods are derived via Taylor series technique. Using the order conditions, eight algebraic order method is presented. Absolute stability of the method is analyzed and the stability region presented. Numerical experiment is conducted on some test problems. Results from the experiment show that the new method is more efficient and accurate than the existing Runge-Kutta and hybrid methods with similar number of function evaluation.

Keywords: Hybrid methods, higher order ODEs, order conditions, numerical methods, stability.

1.   Introduction

Higher order equations (ODEs) are used to model physical phenomena in different areas of applied science, which includes elasticity, fluid mechanics, physics, quantum mechanics and engineering. Only a few of the equations can be solved analytically, as pointed out in Hussain et al. (2016), Ken et al. (2008) and the references therein. Hence, the construction of numerical methods to approximate their solutions become necessary. In view of this, researchers and scholar in the field of numerical analysis contributed immensely in the construction and derivation of several methods for the solution of this class of equations (see Butcher (2008), Dormand (1996), Hairer et al. (1993), Lambert (1991), Langkah et al. (2012), Majid et al. (2010), Mechee et al. (2013), Mechee and Kadhim (2016a)), where the equations are first transformed into systems of first order equations, because the methods are strictly for solving first order equations. The methods, though accurate, have efficiency issues associated with them due to the transformation of the fourth order equations they require. As a result, several direct integrators are proposed. These include cubic spline collocation tau method, see Taiwo and Ogunlaran (2008), logarithmic collocation method, Awoyemi (2005), cubic spline method for fourth order obstacle problems Al-Said et al. (2006), fourth order initial and boundary value problems integrators, Jator (2008). Other such methods can be found in (Hussain et al. (2016), Kayode (2008), Waeleh et al. (2011)) and the references therein.

The general form of the problem considered in this paper is

$$y^{iv}(x) = f(x, y(x)), \ \ y(x_0) = y_0, \ \ y'(x_0) = y'_0, \ \ y''(x_0) = y''_0, \ \ y'''(x_0) = y'''_0, \quad (1)$$

where $y \in R^r$, $f : R \times R^r \to R^r$ is a continuous vector value function. The fact that f is independent of y', y'', y''' explicitly makes (1) special. Typical example of (1) is the ill-posed problem of a beam on elastic foundation, which finds an important engineering application. This problem has been studied in Dong et al. (2014) and Hussain et al. (2016).

In recent time, Runge-Kutta type methods that integrate special third order equations directly are proposed, see for instance Hussain et al. (2017), You and Chen (2013) and the references therein. This motivated Runge-Kutta type methods that integrate (1) directly, which can be found in Hussain et al. (2016), Mechee and Kadhim (2016b) and the references therein. These methods require approximation of three additional points of y'_n, y''_n and y'''_n at each step of the integration. This constitutes computational efficiency issue as the approximation of y_n depends on the derivatives. At this point, we are motivated to derive an integrator that is multistage in nature like the direct Runge-Kutta methods mentioned above, which does not require approximation of y'_n, y''_n and

y_n''' at all, like those proposed in Jikantoro et al. (2018b) for special third order ODEs. Although the method is not self staring, it requires approximation of back values to start the integration like most of the linear multistep methods. The combined properties of multiple stage and multiple step give the method the name hybrid method.

The remaining part of the paper is organized as follows: we present derivation of algebraic order conditions of the method via Taylor series in section 2. The explicit eight algebraic order method is presented in section 3. Absolute stability analysis of the method is presented in section 4. Numerical experiment and discussion is presented in section 5. And conclusion is given in section 6.

2.   Derivation of the Hybrid Method

2.1   The Proposed Method

Define s-stage Runge-Kutta method by

$$Y_i = y_n + h \sum_{j=1}^{s} a_{i,j} f(x_n + c_j h, Y_j), i = 1, ..., s,$$

$$y_{n+1} = y_n + h \sum_{j=1}^{s} b_i f(x_n + c_i h, Y_i). \tag{2}$$

Apply (2) to (1) Jikantoro et al. (2018a), we get

$$Y_i = y_n + h \sum_{j=1}^{s} a_{i,j} U_j, \quad U_i = y_n' + h \sum_{j=1}^{s} a_{i,j} V_j, \quad V_i = y_n'' + h \sum_{j=1}^{s} a_{i,j} W_j,$$

$$W_i = y_n''' + h \sum_{j=1}^{s} a_{i,j} f(x_n + c_j h, Y_j), \quad y_{n+1} = y_n + h \sum_{i=1}^{s} b_i U_i,$$

$$y_{n+1}' = y_n' + h \sum_{i=1}^{s} b_i V_i, \quad y_{n+1}'' = y_n'' + h \sum_{i=1}^{s} b_i W_i, \quad y_{n+1}''' = y_n''' + h \sum_{i=1}^{s} b_i f(x_n + c_i h, Y_i).$$

Eliminate U_i, V_i and W_i from the above equations , we get

$$
\begin{aligned}
Y_i \;=\;& y_n + h\sum_{j=1}^{s} a_{i,j}y_n' + h^2\sum_{j,k=1}^{s} a_{i,j}a_{j,k}y_n'' + h^3\sum_{j,k,l=1}^{s} a_{i,j}a_{j,k}a_{k,l}y_n''' + \\
& h^4\sum_{j,k,l,m=1}^{s} a_{i,j}a_{j,k}a_{k,l}a_{l,m}f(x_n+c_mh,Y_m),
\end{aligned}
$$

$$
\begin{aligned}
y_{n+1} \;=\;& y_n + h\sum_{i=1}^{s} b_iy_n' + h^2\sum_{i,j=1}^{s} b_ia_{i,j}y_n'' + h^3\sum_{i,j,k=1}^{s} b_ia_{i,j}a_{j,k}y_n''' + \\
& h^4\sum_{i,j,k,l=1}^{s} b_ia_{i,j}a_{j,k}a_{k,l}f(x_n+c_lh,Y_l),
\end{aligned}
$$

$$
y_{n+1}' \;=\; y_n' + h\sum_{i=1}^{s} b_iy_i'' + h^2\sum_{i,j=1}^{s} b_ia_{i,j}y_n''' + h^3\sum_{i,j,k=1}^{s} b_ia_{i,j}a_{j,k}f(x_n+c_kh,Y_k),
$$

$$
y_{n+1}'' \;=\; y_n'' + h\sum_{i=1}^{s} b_iy_n''' + h^2\sum_{i,j=1}^{s} b_ia_{i,j}f(x_n+c_jh,Y_j),
$$

$$
y_{n+1}''' \;=\; y_n''' + h\sum_{i=1}^{s} b_if(x_n+c_ih,Y_i).
$$

Assuming that

$$
\sum_{j=1}^{s} a_{i,j} = c_i,\ \sum_{j,k=1}^{s} a_{i,j}a_{j,k} = \frac{c_i^2+c_i}{2},\ \sum_{j,k,l=1}^{s} a_{i,j}a_{j,k}a_{k,l} = \frac{c_i^3+c_i^2+2c_i}{6},
$$

$$
\sum_{i=1}^{s} b_i = \sum_{i,j=1}^{s} b_ia_{i,j} = \sum_{i,j,k=1}^{s} b_ia_{i,j}a_{j,k} = 1,\ i=1,...,s,
$$

$$
\sum_{k,l,m=1}^{s} a_{i,k}a_{k,l}a_{l,m}a_{m,j} = \hat{a}_{i,j},\ \sum_{j,k,l=1}^{s} b_ja_{j,k}a_{k,l}a_{l,i} = \hat{b}_i.
$$

Substitute finite difference formulae for the derivatives, we obtained the proposed method as

$$
y_{n+1} = 4y_n - 6y_{n-1} + 4y_{n-2} - y_{n-3} + h^4\left(\mathbf{b}^T \otimes \mathbf{I}\right)f(x_n+\mathbf{c}h,\mathbf{Y}),
$$

$$
\mathbf{Y} = \mathbf{B}y_n - \mathbf{C}y_{n-1} + \mathbf{D}y_{n-2} - \mathbf{E}y_{n-3} + h^4\left(\mathbf{A} \otimes \mathbf{I}\right)f(x_n+\mathbf{c}h,\mathbf{Y}), \qquad (3)
$$

where

$$
\mathbf{B} = \frac{1}{6}\mathbf{c}^3 + \mathbf{c}^2 + \frac{11}{6}\mathbf{c} + \mathbf{e},\, \mathbf{C} = \frac{1}{6}\mathbf{c}^3 + \frac{5}{6}\mathbf{c}^2 + \mathbf{c},\, \mathbf{D} = \frac{1}{6}\mathbf{c}^3 + \frac{2}{3}\mathbf{c}^2 + \frac{1}{2}\mathbf{c},
$$

$$\mathbf{E} = \frac{1}{6}\mathbf{c}^3 + \frac{1}{2}\mathbf{c}^2 + \frac{1}{3}\mathbf{c} \, , \mathbf{b} = [\hat{b}_1, ..., \hat{b}_m]^T \, , \mathbf{c} = [c_1, ..., c_m]^T \, , \mathbf{e} = [1, ..., 1]^T,$$

$$\mathbf{A} = [\hat{a}_{i,j}] \, , \mathbf{Y} = [Y_1, ..., Y_m]^T$$

and $\mathbf{I}$ is an $m \times m$ dimension identity matrix. The Table 1 below shows the general coefficients of the method.

Table 1: General coefficients of the method

-3	0	0	0	0	0	$\cdots$	0
-2	0	0	0	0	0	$\cdots$	0
-1	0	0	0	0	0	$\cdots$	0
0	0	0	0	0	0	$\cdots$	0
c_5	$\hat{a}_{5,1}$	$\hat{a}_{5,2}$	$\hat{a}_{5,3}$	$\hat{a}_{5,4}$	$\hat{a}_{5,5}$	$\cdots$	$\hat{a}_{5,m}$
c_6	$\hat{a}_{6,1}$	$\hat{a}_{6,2}$	$\hat{a}_{6,3}$	$\hat{a}_{6,4}$	$\hat{a}_{6,5}$	$\cdots$	$\hat{a}_{6,m}$
c_7	$\hat{a}_{7,1}$	$\hat{a}_{7,2}$	$\hat{a}_{7,3}$	$\hat{a}_{7,4}$	$\hat{a}_{7,5}$	$\cdots$	$\hat{a}_{7,m}$
c_8	$\hat{a}_{8,1}$	$\hat{a}_{8,2}$	$\hat{a}_{8,3}$	$\hat{a}_{8,4}$	$\hat{a}_{8,5}$	$\cdots$	$\hat{a}_{8,m}$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
c_m	$\hat{a}_{m,1}$	$\hat{a}_{m,2}$	$\hat{a}_{m,3}$	$\hat{a}_{m,4}$	$\hat{a}_{m,5}$	$\cdots$	$\hat{a}_{m,m}$
	$\hat{b}_1$	$\hat{b}_2$	$\hat{b}_3$	$\hat{b}_4$	$\hat{b}_5$	$\cdots$	$\hat{b}_m$

2.2   Derivation of the order conditions of the Hybrid Method

In this subsection, we derive order conditions of the method. Order condition is a certain relationship between coefficients of a method that causes successive terms in a Taylor series expansion of local truncation error to vanish, Coleman (2003).

To derive the order conditions of the HHM method, we shall consider autonomous case of (1) for simplicity, sine both autonomous and non autonomous forms have the same numerical solution, as shown in You and Chen (2013), and re-write eqn. (3) as follows:

$$\begin{aligned} y_{n+1} &= y_n + \phi(h; y_n), \\ k_i &= f(Y_i), \end{aligned} \qquad (4)$$

where

$$\phi(h; y_n) = 3y_n - 6y_{n-1} + 4y_{n-2} - y_{n-3} + h^4 \sum_{j=1}^{s} \hat{b}_i k_i,$$

and k_i is defined in terms of the components of $\mathbf{Y}$ in eqn. (3). Suppose that the exact solution $y(x)$ at point x_{n+1} is defined as

$$y(x_{n+1}) \quad = \quad y(x_n) + \Psi(h; y(x_n)), \tag{5}$$

then the local truncation error d_{n+1} of HHM method can be expressed as

$$d_{n+1} = y(x_{n+1}) - y_{n+1} \quad = \quad \Psi(h; y(x_n)) - \phi(h; y_n), \tag{6}$$

provided that the local assumption $y(x_n) = y_n$ holds. The next task is to obtain Taylor expansion for both Ψ and ϕ. The Taylor expansion of the two quantities are given below in terms of elementary differential (F).

$$\Psi \quad = \quad hy' + \frac{h^2}{2}y'' + \frac{h^3}{6}y''' + \frac{h^4}{24}F_1^{(4)} + \frac{h^5}{120}F_1^{(5)} + O\left(h^6\right), \quad \phi = hy' + \frac{h^2}{2}y'' + \frac{h^3}{6}y''' +$$

$$\frac{h^3}{24}\left(24\sum_{i=1}^{s}\hat{b}_i - 23\right)F_1^{(4)} + \frac{h^5}{120}\left(120\sum_{i=1}^{s}\hat{b}_ic_i + 121\right)F_1^{(5)} + O\left(h^6\right), \tag{7}$$

where

$$F_1^{(4)} = f, \quad F_1^{(5)} = f_y\left(y'\right), \quad F_1^{(6)} = f_{yy}\left(y', y'\right) + f_y\left(y''\right), \text{e.t.c.}$$

Substituting eqns. (7) into (6) gives Taylor series expansion of the local truncation error of HHM method as follows:

$$t_{n+1} = \left[\frac{h^3}{24}\left(24\sum_{i=1}^{s}\hat{b}_i - 24\right)F_1^{(4)} + \frac{h^5}{120}\left(120\sum_{i=1}^{s}\hat{b}_ic_i + 120\right)F_1^{(5)} + O(h^6)\right]. \tag{8}$$

Hence, the algebraic order conditions of the HHM method up to order eight are summarized in Table 2.

3. Derivation of Explicit HHM8

In this section we derive an explicit HHM method with eight algebraic order using the algebraic order conditions derived in section 2 above. To do that, the equations of the order conditions up to order eight are solved.

Substituting the values of c_i, $i = 1, ...4$ from Table 1 above leaves us with eighteen equations to be solved in twenty five unknown parameters. The coefficients of the method obtained (after solving the equations and choosing the free parameters at random) are summarized in the table below and the method is denoted by HHM8.

Table 2: Order Conditions of HHM

Order	condition
1	$\sum \hat{b}_i = 1$
2	$\sum \hat{b}_i c_i = -1$
3	$\sum \hat{b}_i c_i^2 = \frac{4}{3}$
4	$\sum \hat{b}_i c_i^3 = -2$
5	$\sum \hat{b}_i c_i^4 = \frac{33}{10}, \ \sum \hat{b}_i c_i \hat{a}_{i,j} = -\frac{1}{720}$
6	$\sum \hat{b}_i c_i^5 = -\frac{35}{6}, \ \sum \hat{b}_i \hat{a}_{i,j} c_j = -\frac{1}{720}$ $\sum \hat{b}_i c_i \hat{a}_{i,j} = \frac{1}{720}$
7	$\sum \hat{b}_i c_i^6 = \frac{229}{21}, \ \sum \hat{b}_i \hat{a}_{i,j} c_j^2 = -\frac{1}{840}$ $\sum \hat{b}_i c_i \hat{a}_{i,j} c_j = \frac{17}{5040}, \ \sum \hat{b}_i c_i^2 \hat{a}_{i,j} = \frac{43}{5040}$
8	$\sum \hat{b}_i c_i^7 = -\frac{229}{21}, \ \sum \hat{b}_i c_i^2 \hat{a}_{i,j} c_j = -\frac{1}{80}$ $\sum \hat{b}_i c_i^3 \hat{a}_{i,j} = -\frac{19}{1680}, \ \sum \hat{b}_i c \hat{a}_{i,j} c_j^3 = \frac{1}{1260}$ $\sum \hat{b}_i c_i \hat{a}_{i,j} c_j^2 = -\frac{7}{1080}$

Table 3: Coefficients of HHM8

-3	0	0	0	0	0	0	0
-2	0	0	0	0	0	0	0
-1	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0
$\frac{1}{27}$	$-\frac{3951662}{94143178827}$	$\frac{150621823}{62762119218}$	$\frac{223863772}{31381059609}$	$\frac{77032153}{188286357654}$	0	0	0
$\frac{1}{7}$	$\frac{2131111081}{680811468498}$	$-\frac{4680788518}{113468578083}$	$\frac{2804326863}{45387431232}$	$-\frac{90199966406}{340405734249}$	$\frac{1}{4}$	0	0
$-\frac{6121}{4179}$	$-\frac{732342756543089477586897905}{12200184546025990576061857617 9}$	$\frac{1493386649887374823957463756}{406672818200866635255539525393}$	$-\frac{1492256617569457896820650683}{406672818200866635255539525393}$	$\frac{62469989485733548263762351404}{12200184546025990576061857617 9}$	$-\frac{1}{2}$	$\frac{1}{58}$	0
	$\frac{1173167}{1041701760}$	$\frac{111715}{885852}$	$\frac{6172283}{13050240}$	$\frac{4224601}{1101780}$	$-\frac{126686499903}{28530187840}$	$\frac{180893741}{212826240}$	$\frac{50474033141187348817299}{3236845831327368040910720}$

4. Stability Analysis

Consider the fourth order scalar test equation

$$y^{iv} = -\lambda^4 y, \quad \lambda > 0. \tag{9}$$

To analyze the absolute stability of the proposed method, (3) is applied to (9) as follows: the second component of (3) gives

$$\mathbf{Y}\left(\mathbf{I} + z\mathbf{A}\right) = \frac{1}{6}\mathbf{B}y_n - \frac{1}{2}\mathbf{C}y_{n-1} + \frac{1}{2}\mathbf{D}y_{n-2} - \frac{1}{6}\mathbf{E}y_{n-3},$$

$$\mathbf{Y} = \left(\frac{1}{6}\mathbf{B}y_n - \frac{1}{2}\mathbf{C}y_{n-1} + \frac{1}{2}\mathbf{D}y_{n-2} - \frac{1}{6}\mathbf{E}y_{n-3}\right)\left(\mathbf{I} + z\mathbf{A}\right)^{-1},$$

where $z = (\lambda h)^4$. And for the first component, we get

$$y_{n+1} = \left(4\mathbf{e} - z\frac{1}{6}\mathbf{b}^T\mathbf{B}\left(\mathbf{I} + z\mathbf{A}\right)\right) y_n - \left(6\mathbf{e} - z\frac{1}{2}\mathbf{b}^T\mathbf{C}\left(\mathbf{I} + z\mathbf{A}\right)\right) y_{n-1} +$$

$$\left(4\mathbf{e} - z\frac{1}{2}\mathbf{b}^T\mathbf{D}\left(\mathbf{I} + z\mathbf{A}\right)\right) y_{n-2} - \left(\mathbf{e} - z\frac{1}{6}\mathbf{b}^T\mathbf{E}\left(\mathbf{I} + z\mathbf{A}\right)\right) y_{n-3},$$

$$\chi(\xi) = \xi^4 - \mathbf{K}_1\xi^3 + \mathbf{K}_2\xi^2 - \mathbf{K}_3\xi + \mathbf{K}_4, \tag{10}$$

where K_i, $i = 1, 2, 3, 4$ are variables that depend on the coefficients of the method and z. Eqn. (10) is the stability polynomial of the proposed method. An interval $(-z_a, \alpha)$ is said to be interval of absolute stability of HHM method if, $\forall\ z \in (-z_a, \alpha)$, $|\xi_{1,2,3,4}| < 1$, where $\xi_{1,2,3,4}$ are roots of eqn. (10). Absolute stability region is a region enclosed by the set of points for which $|\xi| = 1$. The region is easily obtained by putting $\xi = e^{I\theta}$ in eqn. (10) for $0 \le \theta \le 2\pi$, solve for z, then map out the boundary using MAPLE. The shaded portion in Figure 1 below is the stability region of the proposed method.
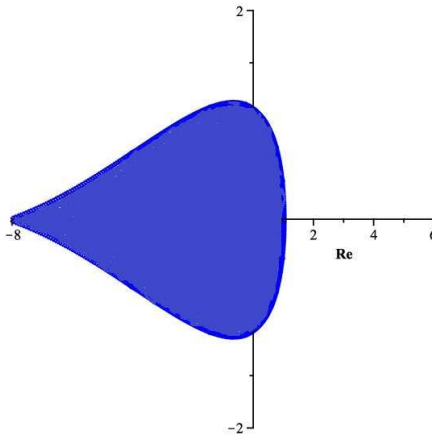

Figure 1: Stability region of HHM8

4.1 Zero Stability

The HHM8 method is said to be zero stable if the roots ϑ_j, $j = 1, 2, 3, 4$, of the first characteristics polynomial $\chi(\vartheta)$, defined by

$$\chi(\vartheta) = \sum_{i=0}^{4} \gamma_i \vartheta^i,$$

satisfy $|\vartheta_j| \leq 1$, $j = 1, 2, 3, 4$ and for the roots with $|\vartheta_j| = 1$, the multiplicity does not exceed 4.

4.2 Consistency

The HHM8 method is said to be consistent if it has order greater than one.

The first characteristics polynomial of the method is

$$\chi(\vartheta) = \xi^4 - 4\vartheta^3 + 6\vartheta^2 - 4\vartheta + 1 = 0,$$

which implies that $\vartheta = 1$ four times. Therefore, the HHM8 method is zero stable. We also note that, provided the order conditions are satisfied, the order of the method is greater than one, which implies that it's consistent. Hence, we conclude that the HHM8 method is convergent.

5. Implementation

In this section, we present numerical results of the proposed method and some existing methods as they are applied to some test problems that are listed below. Efficiency as well as accuracy of the methods are measured by plotting the $\log_{10}$ of maximum errors recorded with different step lengths h in a given interval [a,b] against computation efforts measured by total number of function call for each method.

- **HHM8**: the proposed eight order four-stage explicit hybrid method derived in this paper;

- **RKFD**: fifth order explicit four-stage Runge-Kutta direct method derived in Hussain et al. (2016);

- **JHM**: fifth order four-stage explicit hybrid method for second order oscillatory problems derived in Jikantoro et al. (2016);

- **FHM**: third order three-stage explicit hybrid method for second order oscillatory problems derived in Franco (2006);

- **ME**: logarithm to base 10 of maximum absolute error;

- **FE**: logarithm to base 10 of total function call.

5.1 Test problems

Problem 1:
$$y'''' = -4y, \quad 0 \le x \le 5,$$
$$y(0) = 1, \quad y'(0) = 1, \quad y''(0) = 2, \quad y'''(0) = 2,$$
$$y(x) = \exp(x)\sin(x). \qquad \text{Source: Mechee and Kadhim (2016b)}$$

Problem 2:
$$y'''' = y^2 + \cos^2(x) + \sin(x) - 1, \quad 0 \le x \le 5,$$
$$y(0) = 0, \quad y'(0) = 1, \quad y''(0) = 0, \quad y'''(0) = -1,$$
$$y(x) = \sin(x). \qquad \text{Source: Hussain et al. (2016)}$$

Problem 3:
$$y'''' = \frac{3\sin(y)\left(3 + 2\sin^2(y)\right)}{cos^7(y)}, \quad 0 \le x \le \frac{\pi}{4},$$
$$y(0) = 0, \quad y'(0) = 1, \quad y''(0) = 0, \quad y'''(0) = 1,$$
$$y(x) = \sin^{-1}(x). \qquad \text{Source: Hussain et al. (2016)}$$

Problem 4: the ill-posed problem of a beam on elastic foundation:
$$y'''' = x - y, \quad 0 \le x \le 5,$$
$$y(0) = 0, \quad y'(0) = 0, \quad y''(0) = 0, \quad y'''(0) = 0,$$
$$y(x) = 1 - 1/2\,e^{-1/2\sqrt{2}x}\cos\left(1/2\sqrt{2}x\right) - 1/2\,e^{1/2\sqrt{2}x}\cos\left(1/2\sqrt{2}x\right).$$
$$\text{Source: Jikantoro et al. (2018a)}$$

Problem 5:
$$y_1'''' = e^{3x}y_2, \quad y_1(0) = 1, \quad y_1'(0) = -1, \quad y_1''(0) = 1, \quad y_1'''(0) = -1,$$
$$y_2'''' = 256e^{-x}y_3, \quad y_2(0) = 1, \quad y_2'(0) = -4, \quad y_2''(0) = 16, \quad y_2'''(0) = -64,$$
$$y_3'''' = 81e^{-x}y_4, \quad y_3(0) = 1, \quad y_3'(0) = -3, \quad y_3''(0) = 9, \quad y_3'''(0) = -27,$$
$$y_4'''' = 16e^{-x}y_1, \quad y_4(0) = 1, \quad y_4'(0) = -2, \quad y_4''(0) = 4, \quad y_4'''(0) = -8,$$
$$y_1(x) = e^{-x}, \quad y_2(x) = e^{-4x}, \quad y_3(x) = e^{-3x}, \quad y_4(x) = e^{-2x}, \quad 0 \le x \le 2.$$
$$\text{Source: Dong et al. (2014)}$$

The proposed method is not self-starting and it has similar function evaluation as Runge-Kutta methods. To start the integration with the method, initial point of the solution and three additional points are required as starting values. The starting values are computed here by RKFD, see Hussain et al. (2016). When the starting values are obtained, the integration proceeds with $s - 3$ function calls at every step. That means the proposed method has 4 function evaluation per step.

Note that JHM and FHM are specifically for solving second order equations directly. As such, each of the test equations is transformed to twice its dimension for the two methods to be applied. Figures 1–5 are the graphs showing the accuracy and the computational effort of the HHM8 method as compared with those of the existing methods. Figure 6 shows the stability region of the method. It is observed that for all the test problems solved, the proposed method appear to be more accurate with comparable number of function evaluation. In addition, unlike the RKFD, integration with HHM8 does not require computer memory space to store the values of derivatives of the solution.
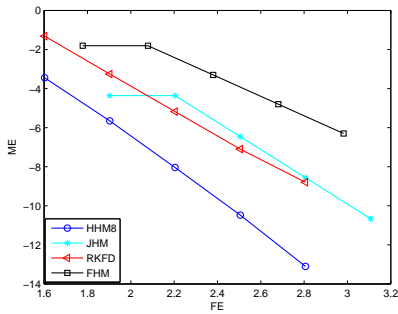
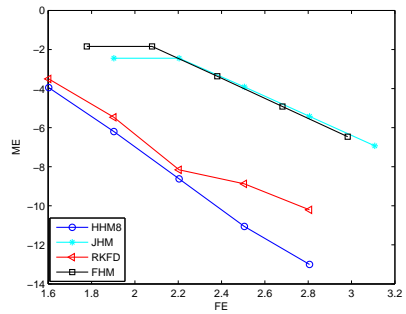Figure 2: Efficiency curves for problem 1, $h = 2^{-i}$, $i = 2, ..., 6$.

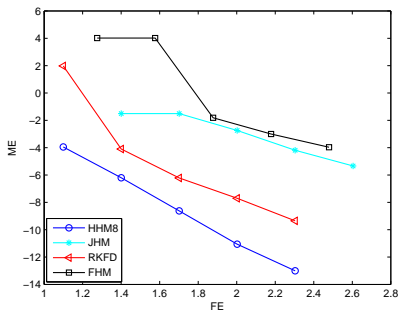Figure 3: Efficiency curves for problem 2, $h = 2^{-i}$, $i = 2, ..., 6$.

Figure 4: Efficiency curves for problem 3, $h = 2^{-i}$, $i = 4, ..., 8$.
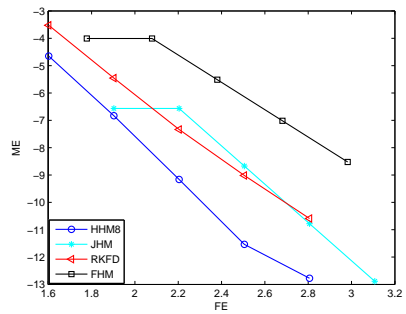

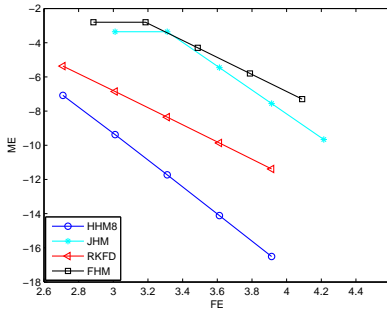Figure 5: Efficiency curves for problem 4, $h = 2^{-i}$, $i = 2, ..., 6$.

Figure 6: Efficiency curves for problem 5, $h = 2^{-i}$, $i = 4, ..., 8$.

6. Conclusion

A hybrid method that directly integrates special class of fourth order equations is proposed and derived. The method is similar to the class of two-step hybrid methods for solving special second order ODEs proposed in Coleman (2003). The major improved difference between the method and the methods of its kind, example RKFD, is that it doesn't require approximation of y', y'' and y''' for the numerical integration. Using Taylor series technique, algebraic order conditions of the method are derived. The order conditions are used to derive HHM8 method. Stability of the method is also analyzed and stability region presented. Numerical results presented in Figures 2-6 reveal that the new method is better than the existing methods considered in the paper.

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References

Al-Said, E. A., Noor, M. A., and Rassias, T. M. (2006). Cubic splines method for solving fourth-order obstacle problems. *Appl. math. comput.*, 174(1):180–187.

Awoyemi, D. O. (2005). Algorithmic collocation approach for direct solution of

fourth-order initial-value problems of ordinary differential equations. *Int. J. of Comput. Math.*, 82(3):321–329.

Butcher, J. C. (2008). *Numerical Methods for Ordinary Differential Equations, second ed.* John Wiley & Sons, England.

Coleman, J. P. (2003). Order conditions for a class of two-step methods for $y'' = f(x, y)$. *IMA J. of Numer. Anal.*, 23:197–220.

Dong, L., Alotaibi, A., Mohiuddine, S. A., and Atluri, S. N. (2014). Computational methods in engineering: a variety of primal & mixed methods, with global & local interpolations, for well-posed or ill-posed bcs. *CMES: Comput. Model. Eng. & Sci.*, 99(1):1–85.

Dormand, J. R. (1996). *Numerical methods for differential equations: a computational approach, second ed.*, volume 3. CRC Press.

Franco, J. M. (2006). A class of explicit two-step hybrid methods for second-order ivps. *J. Comput. Appl. Math.*, 187(1):41–57.

Hairer, E., Norsett, S. P., and Wanner, G. (1993). *Solving Ordinary Differential Equations I: Nonstiff Problems, second ed.* Springer-Verlag, Berlin.

Hussain, K., Ismail, F., and Senu, N. (2016). Solving directly special fourth-order ordinary differential equations using runge–kutta type method. *J. Comput. Appl. Math.*, 306:179–199.

Hussain, K. A., Ismail, F., Senu, N., and Rabiei, F. (2017). Fourth-order improved runge-kutta method for directly solving special third-order ordinary differential equations. *Iranian J. Appl. Sci. Tech.*, 41:429–437.

Jator, S. N. (2008). Numerical integrators for fourth order initial and boundary value problems. *Int. J. Pure Appl. Math.*, 47(4):563–576.

Jikantoro, Y. D., Ismail, F., and Senu, N. (2016). Higher order dispersive and dissipative hybrid method for the numerical solution of oscillatory problems. *Int. J. Comput. Math.*, 93(6):929–941.

Jikantoro, Y. D., Ismail, F., Senu, N., and Ibrahim, Z. B. (2018a). A class of hybrid methods for direct integration of fourth-order ordinary differential equations. *Bull. Malaysian Math. Sci. Soc.*, 41:985–1010.

Jikantoro, Y. D., Ismail, F.and Senu, N., and Ibrahim, Z. B. (2018b). Hybrid methods for direct integration of special third order ordinary differential equations. *Appl. Math. Comput.*, 320:452–463.

Kayode, S. J. (2008). An efficient zero-stable numerical method for fourth-order differential equations. *Int. J. Math. Math. Sci.*, 2008.

Ken, Y. L., Ismail, F., Suleiman, M., and Amin, S. M. (2008). Block methods based on newton interpolations for solving special second order ordinary differential equations directly. *J. Math. Stat.*, 4:174–180.

Lambert, J. D. (1991). *Numerical methods for ordinary differential systems: the initial value problem, third ed.* John Wiley & Sons, Inc.

Langkah, K. B. D.-T. E., Majid, Z. A., Azmi, N. A., Suleiman, M., and Ibrahaim, Z. B. (2012). Solving directly general third order ordinary differential equations using two-point four step block method. *Sains Malaysiana*, 41:623–632.

Majid, Z. A., Suleiman, M., and Azmi, N. A. (2010). Variable step size block method for solving directly third order ordinary differential equations. *Far East J. Math. Sci.*, 41:63–73.

Mechee, M., Ismail, F., Senu, N., and Siri, Z. (2013). Directly solving special second order delay differential equations using runge-kutta-nyström method. *Math. Probl. Eng.*, 2013.

Mechee, M. and Kadhim, M. (2016a). Explicit direct integrators of rk type for solving special fifth-order ordinary differential equations. *American J. Appl. Sci.*

Mechee, M. S. and Kadhim, M. A. (2016b). Direct explicit integrators of rk type for solving special fourth-order ordinary differential equations with an application. *Global J. Pure and Appl. Math.*, 12:4687–4715.

Taiwo, O. A. and Ogunlaran, O. M. (2008). Numerical solution of fourth order linear ordinary differential equations by cubic spline collocation tau method. *J. Math. Stat.*

Waeleh, N., Abdul, M. Z., Ismail, F., and Suleiman, M. (2011). Numerical solution of higher order ordinary differential equations by direct block code. *J. of Math. Stat.*, 8(1):77–81.

You, X. and Chen, Z. (2013). Direct integrators of runge–kutta type for special third-order ordinary differential equations. *Appl. Numer. Math.*, 74:128–150.